



Analytical-Numerical Solution of Thermoelastic Problem in a Semi-Infinite Medium under Green and Naghdi Theory

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Abstract—The present work is aimed at the study the analytical and numerical solutions of thermoelastic problem in a semi-infinite medium in the context of the Green and Naghdi theory of type III. The governing equations are expressed in Laplace transform domain and solved in the domain by analytical method and finite element method. The solutions of the problem in the physical domain are obtained by using a numerical method for the inversion of the Laplace transforms based on Stehfest's method. Numerical results for the temperature distribution, displacement and thermal stress are represented graphically.

Index Terms: Finite element method; Laplace transforms; Green and Naghdi theory

I. INTRODUCTION

THE classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms contrary to the fact that elastic changes produce heat effects. Second, the heat equation is of parabolic type predicting infinite speeds of propagation for heat waves.

Biot [1] introduced the theory of coupled thermoelasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of classical theory. However, both theories share the second shortcoming since the heat equation for the coupled theory is also parabolic. Lord and Shulman [2] introduce the theory of generalized thermoelasticity with one relaxation time. In this theory a modified law of heat

conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this a hyperbolic one and hence, automatically eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. Most often the solution obtained using this theory differ little quantitatively from those obtained using either the coupled or the uncoupled theories, though, the solutions differ quantitatively. However, for many problems involving steep heat gradients and when short time effects are sought, this theory gives markedly different values than those predicted by any of the other theories. This is the case encountered in many problems in industry especially inside nuclear reactors where very high heat gradients act for very short times. The theory of couple thermoelasticity was extended by Green and Lindsay [3] by including the thermal relaxation time in constitutive relations. The theory was extended for anisotropic body by Dhaliwal and Sherief [4]. In the decade of the 1990's Green and Naghdi [5], [6] and [7] proposed three new thermoelastic theories based on entropy equality rather than the usual entropy inequality. The constitutive assumptions for the heat flux vector are different in each theory. Thus, they obtained three theories that they called thermoelasticity of type I, type II and type III, respectively. When the theory of type I is linearized we obtain the classical system of thermoelasticity. The theory of type II (is a limiting case of the type III) does not admit energy dissipation. In the context of the linearized version of this theory, theorems on uniqueness of solutions have been established by Hitnarski and Ignazack [8] and Green and

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Naghdi [7] boundary-initiated waves in a half-space and in unbounded body with cylindrical cavity have been studied by Green and Naghdi [5], Chandrasekharaiah and Srinath [9] and [10]. The effect of rotation on the reflection of magneto-thermoelastic waves under thermoelasticity without energy dissipation have been studied by Othman and Song [11] and [12].

Youssef [13-14] has formulated a problem of an infinite body having a cylindrical cavity using the Laplace transform technique and the same has been solved numerically based on the Fourier series expansion. Abbas and Abbas et. al [15-21] applied the finite element method in different problems.

In the present paper, we have considered a thermal shock problem in a semi-infinite medium under Green and Naghdi theory of type III by analytical method (Exact solution) and numerical method (finite element method). Numerical results for the temperature distribution, displacement and thermal stress are represented graphically. Finally, the accuracy of the finite element formulation was validated by comparing the analytical and numerical solutions for the field quantities.

II. BASIC EQUATION AND FORMULATION

For a linear, homogenous and isotropic thermoelastic continuum, the generalized field equations can be presented in a unified form as [13]

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) u_{j,ji} + \mu u_{i,jj} - \gamma T_{,i} + F_i \quad (1)$$

$$K_{ij}^* T_{,ii} + K_{ij} \dot{T}_{,ii} = \rho c_e \ddot{T} + \gamma T_0 \ddot{u}_{j,j} \quad (2)$$

The constitutive equations are given by

$$\tau_{ij} = \lambda u_{i,j} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) - \gamma (T - T_0) \delta_{ij} \quad (3)$$

Where λ and μ are Lamé's constants, ρ is the density of medium, c_e is specific heat at constant strain, t is the time, T is the temperature, T_0 is the reference temperature, K_{ij} are the thermal conductivity, K_{ij}^* are the material constant characteristic of the theory, δ_{ij} is the Kronecker symbol, τ_{ij} are the components of stress tensor, u_i are the components of displacement vector, F_i are the body force vector, $\gamma = (3\lambda + 2\mu)\alpha_t$, α_t is the coefficient of linear thermal expansion. It assumed that the state of the medium depends only on x and the time variable t . It is assumed that there are no body forces and heat sources in the medium and that the

plane $x=0$ is taken to be traction free. Thus the field equations (1)-(3) in a one-dimensional case can be put as

$$(\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial T}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \quad (4)$$

$$K^* \frac{\partial^2 T}{\partial x^2} + K \frac{\partial^3 T}{\partial t \partial x^2} = \frac{\partial^2}{\partial t^2} \left(\rho c_e T + \gamma T_0 \frac{\partial u}{\partial x} \right), \quad (5)$$

$$\tau_{xx} = (\lambda + 2\mu) \frac{\partial u}{\partial x} - \gamma (T - T_0). \quad (6)$$

For convenience, we shall use the following non-dimensional variables:

$$(x^\circ, u^\circ) = \frac{1}{c\eta} (x, u), \quad T^\circ = \frac{T - T_0}{T_0}, \quad t^\circ = \frac{1}{\eta} t, \quad \tau_{xx}^\circ = \frac{\tau_{xx}}{\lambda + 2\mu},$$

where $c^2 = \frac{\lambda + 2\mu}{\rho}$ and $\eta = \frac{K}{\rho c_e c^2}$. Into equations (4)-(6),

one may obtain (after dropping the superscript $^\circ$ for convenience)

$$\frac{\partial^2 u}{\partial x^2} - \zeta \frac{\partial T}{\partial x} = \frac{\partial^2 u}{\partial t^2}, \quad (7)$$

$$\varepsilon_1 \frac{\partial^2 T}{\partial x^2} + \frac{\partial^3 T}{\partial t \partial x^2} = \frac{\partial^2}{\partial t^2} \left(T + \varepsilon_2 \frac{\partial u}{\partial x} \right), \quad (8)$$

$$\tau_{xx} = \frac{\partial u}{\partial x} - \zeta T, \quad (9)$$

where $\zeta = \frac{T_0 \gamma}{\lambda + 2\mu}$, $\varepsilon_1 = \frac{k^*}{\rho c_e c^2}$, $\varepsilon_2 = \frac{\gamma}{\rho c_e}$.

The non-dimensional forms of the initial and boundary condition are:

$$u(x, 0) = \frac{\partial u(x, 0)}{\partial t} = 0, \quad T(x, 0) = \frac{\partial T(x, 0)}{\partial t} = 0, \quad (10)$$

$$\tau_{xx}(0, t) = 0, \quad T(0, t) = T_1 H(t), \quad (11)$$

where $H(t)$ denotes the Heaviside unit step function.

III. BASIC EQUATIONS IN THE LAPLACE TRANSFORM DOMAIN

Applying the Laplace transform for equations (7)-(9) define by the formula

$$\bar{f}(s) = L[f(t)] = \int_0^\infty f(t) e^{-st} dt. \quad (12)$$

Hence, we obtain the following system of differential equations



$$\frac{d^2 \bar{u}}{dx^2} - g_2 \frac{d\bar{T}}{dx} - g_1 \bar{u} = 0, \quad (13)$$

$$\frac{d^2 \bar{T}}{dx^2} - g_3 \bar{T} - g_4 \frac{d\bar{u}}{dx} = 0, \quad (14)$$

$$\bar{\tau}_{xx} = \frac{d\bar{u}}{dx} - \zeta \bar{T}, \quad (15)$$

$$\text{with } g_1 = s^2, g_2 = \frac{T_0 \gamma}{\lambda + 2\mu}, g_3 = \frac{s^2}{\epsilon_1 + s} \text{ and } g_4 = \frac{s^2 \epsilon_2}{\epsilon_1 + s}, \text{ and } s$$

denotes the Laplace transform parameter. All the state functions initially are equal to zero, and the boundary conditions (11) become

$$\bar{\tau}_{xx}(0, s) = 0, \bar{T}(0, s) = \frac{T_1}{s}. \quad (16)$$

IV. EXACT SOLUTION

Eliminating \bar{T} from the equations (13) and (14) we get

$$\left(\frac{d^4}{dx^4} - (g_1 + g_3 + g_2 g_4) \frac{d^2}{dx^2} + g_1 g_3 \right) \bar{u} = 0, \quad (17)$$

The solutions of Equation (17) bounded at infinity can be written in the form:

$$\bar{u} = A_1 e^{-m_1 x} + A_2 e^{-m_2 x}, \quad (18)$$

where A_1 and A_2 are parameters depending on s to be determined from the boundary conditions, m_1 and m_2 are the roots with positive real parts of the characteristic equation

$$m^4 - (g_1 + g_3 + g_2 g_4) m^2 + g_1 g_3 = 0, \quad (19)$$

m_1 and m_2 are given by

$$m_1 = \sqrt{\frac{1}{2} \left[g_1 + g_3 + g_2 g_4 + \sqrt{(g_1 + g_3 + g_2 g_4)^2 - 4g_1 g_3} \right]},$$

$$m_2 = \sqrt{\frac{1}{2} \left[g_1 + g_3 + g_2 g_4 - \sqrt{(g_1 + g_3 + g_2 g_4)^2 - 4g_1 g_3} \right]},$$

From equation (18) into equations (13) and (14), the expression for temperature can be written in the form

$$\bar{T} = B_1 e^{-m_1 x} + B_2 e^{-m_2 x}, \quad (20)$$

$$\text{where } B_i = E_i A_i, E_i = \frac{g_1 + g_2 g_4 - m_i^2}{g_2 g_3} m_i, \quad i = 1, 2.$$

Substituting from equations (19) and (20) into equation (15), we obtain

$$\bar{\tau}_{xx} = g_1 A_1 e^{-m_1 x} + g_1 A_2 e^{-m_2 x}. \quad (21)$$

V. NUMERICAL SOLUTION

In order to investigate the thermoelastic interactions in a semi-infinite medium without energy dissipation, the finite element method (FEM) [22] is adopted due to its flexibility in modeling layered structures and its capability in obtaining full field numerical solution. The governing equations (13) and (14) are coupled with boundary conditions (16). The numerical values of the dependent variables like displacement \bar{u} and the temperature \bar{T} are obtained at the interesting points which are called degrees of freedom. The weak formulations of the non-dimensional governing equations are derived. We assume that the master element has its local coordinates in the range $[-1, 1]$. In our case, the one-dimensional quadratic elements are used, which given by: Linear shape functions

$$N_1 = \frac{1}{2}(1 - \xi), \quad N_2 = \frac{1}{2}(1 + \xi),$$

Quadratic shape functions

$$N_1 = \frac{1}{2}(\xi^2 - \xi), \quad N_2 = 1 - \xi^2, \quad N_3 = \frac{1}{2}(\xi^2 + \xi),$$

VI. NUMERICAL INVERSION OF THE LAPLACE TRANSFORMS

For the final solution of temperature, displacement and stress distributions in the time domain, we adopt a numerical inversion method based on the Stehfest [23]. In this method, the inverse $f(t)$ of the Laplace transform $f(s)$ is approximated by the relation

$$f(t) = \frac{\ln 2}{t} \sum_{i=1}^n V_i F\left(\frac{\ln 2}{t} i\right), \quad (26)$$

Where V_i is given by the following equation:

$$V_i = (-1)^{\binom{n}{2}+1} \sum_{k=\frac{i+1}{2}}^{\min\left(i, \frac{n}{2}\right)} \frac{k^{\binom{n}{2}+1} (2k)!}{\left(\frac{n}{2} - k\right)! k! (i-k)! (2k-1)!}. \quad (27)$$

The parameter n is the number of terms used in the summation in equation (26) and should be optimized by trial and error. Increasing n increases the accuracy of the result up to a point, and then the accuracy declines because of increasing round-off

errors. An optimal choice of $10 \leq n \leq 14$ has been reported by Lee et al. for some problem of their interest [24].

VII. NUMERICAL RESULTS AND DISCUSSION

In order to illustrate the problem, the copper material was chosen for purposes of numerical evaluations. The physical data which given as [13]

$$\lambda = 7.76 \times 10^{10} (\text{kg})(\text{m})^{-1}(\text{s})^{-2},$$

$$\mu = 3.86 \times 10^{10} (\text{kg})(\text{m})^{-1}(\text{s})^{-2},$$

$$T_0 = 293 (\text{K}), T_1 = 1,$$

$$K = 3.68 \times 10^2 (\text{kg})(\text{m})(\text{K})^{-1}(\text{s})^{-3},$$

$$c_e = 3.831 \times 10^2 (\text{m})^2 (\text{K})^{-1}(\text{s})^{-2},$$

$$\rho = 8.954 \times 10^3 (\text{kg})(\text{m})^{-3},$$

$$\alpha_t = 17.8 \times 10^{-6} (\text{K})^{-1}.$$

Here all the variables/parameters are taken in non-dimensional forms. The results for displacement, temperature and stress has been carried out by taking $T_1 = 1$. Figures 1, 2 and 3 exhibit the variation of the displacement, temperature and stress with space x for different values of time ($t = 0.1, 0.2, 0.3, 0.4$). It is obvious from figure 1 that the displacement is negative at $x=0$ where its magnitude is maximum. The displacement increases from the negative value to a positive value. In the positive values, the displacement has a peak value that depends on the values of the time. It is obvious from figure 2 that the temperature decreases with the increase of the space but they increase when increasing the time. It is obvious from figure 3 which gives the stress variation at different instants of time with the space. Its magnitude increases from zero to a maximum value after that decreases rapidly as x increases.

Finally, figures 1-3 illustrates the solution obtained numerically by finite element method (—○—) overlaid onto the solution obtained analytically (——). The accuracy of the finite element formulation was validated by comparing the analytical and numerical solutions for the field quantities.

VIII. CONCLUSION

In our investigation, a solution of thermal shock problem of generalized thermoelasticity in a homogeneous isotropic

semi-infinite medium under Green and Naghdi theory of type III was presented. The problem has been solved the analytical method (exact solution) and numerically using the finite element method. The grid size has been refined and consequently the values of different parameters. Further refinement of mesh size over 6000 elements does not change the values considerably, which is therefore accepted as the grid size for computing purpose. The accuracy of the finite element formulation was validated by comparing the analytical and numerical solutions for the field quantities.

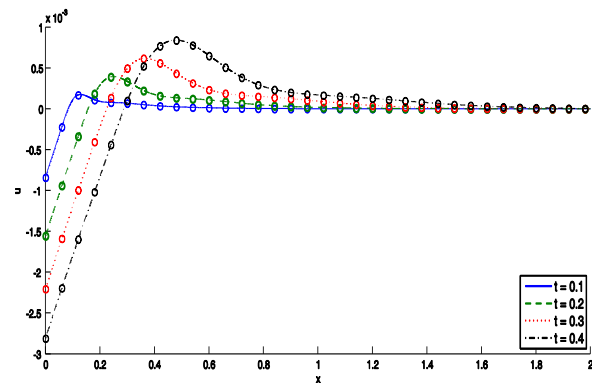


Figure 1. Variation of displacement

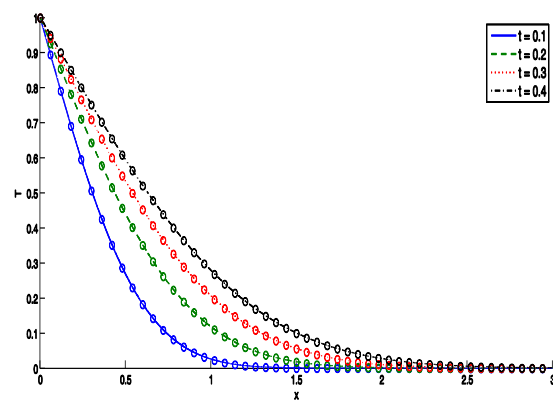


Figure 2. Variation of temperature

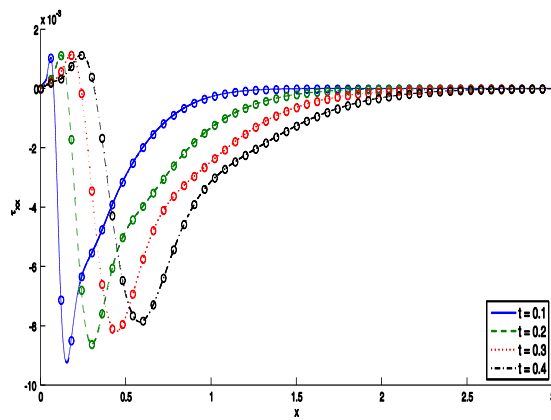


Figure 3. Variation of stress

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